Finite-size effects in diffusion-limited aggregation

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This paper discusses a large variety of numerical results on diffusion-limited aggregation (DLA) to support the view that asymptotically large DLA is self-similar and the scaling of the geometry can be specified by the fractal dimension alone. Deviations from simple scaling observed in many simulations are due to finite-size effects. I explain the relationship between the finite-size effects in various measurements and how they can arise due to a crossover of the noise magnitude in the growth process. Complex scaling hypotheses including anomalous scaling of the width of the growing region, multiscaling of the cluster radial density, infinite drift of the ϵ-neighborhood filling ratio, nonmultifractal scaling of the growth probability measure, and geometrical multifractality, are shown to lead to physically unacceptable predictions.

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I. INTRODUCTION

Diffusion-limited aggregation (DLA) [1] is one of the most intensively studied fractal growth models [2]. The structure and properties of asymptotically large off-lattice aggregates are of central importance to the theoretical understanding of DLA growth. However, although clusters containing more than 10^6 particles have been used in many studies, great controversies still exist on various asymptotic properties of DLA.

DLA was originally suggested to be a self-similar fractal. The scaling of the structure is expected to be characterized solely by the fractal dimension D, which can be measured from the scaling relation

$$N \sim R_g^D \tag{1}$$

where N and R_g are the cluster size and the radius of gyration, respectively. Several theories of DLA based on the self-similarity hypothesis were suggested. For instance, Pietronero, Erzan, and Evertsz [3] analyzed the growth using a fixed scale transformation closely related to real space renormalization techniques. Halsey and Leibig [4] studied a binary hierarchical model of DLA with emphasis on the competition between branches emanating from the same point. Blumenfeld [5] studied the growth by considering the statistical properties of the conformal mapping from the unit disk to the cluster. In each of these approaches, the growth mechanism can be coarse grained to some nontrivial fixed point and these theories are all consistent with simple scaling of DLA.

However, results from subsequent simulations have led to the suggestion of a number of anomalous scaling behaviors. Examples include anomalous scaling of the width of the growing region [6], deviation of the growth probability measure from usual multifractal behavior [2], multiscaling of the radial density [7], geometrical multifractality [8], decreasing lacunarity [9], and so on. These complex scaling hypotheses describe the numerical results on finite clusters better than conventional scaling. Many authors believe that they are also the correct asymptotic descriptions. However, many others, including the present author, think that simple scaling cannot account for some measurements accurately only because of the existence of very strong transients or finite-size effects. For much larger clusters, simple scaling should hold accurately. However, it is extremely difficult, if not impossible, to obtain solid evidence for either view since numerical tests have to be done on finite clusters and, on the other hand, no trustworthy first principle theory on DLA exists. We are thus forced to examine carefully the abundant but less discriminating information in order to resolve the controversies.

This paper aims at supporting the view that DLA follows simple scaling with strong finite-size effects, and *none* of the complex scalings hold asymptotically. It is well established that DLA does have strong finite-size effects in some aspects. For instance, DLA is locally isotropic asymptotically but anisotropy is expected to persist up to a cluster size of about 10^{11} particles [10]. The decay exponent of the correlation in the tangential direction converges extremely slowly to that in the radial direction [11]. Other examples include the finite-size effects on the noise magnitude [12], the branch subtending angle [13], and the radial density [14]. These finite-size effects are closely related to each other and can lead to anomalous results in many other measurements. As a result, *ad hoc* complex scaling hypotheses supported only by a better fit to an individual set of numerical data can well be an illusion caused by finite-size effects. To establish any complex scaling firmly, one has to make sure that it is compatible with some plausible overall picture of DLA growth. Unfortunately, the consequences of many suggested complex scaling behaviors have not been sufficiently explored.

In the following, I will argue that simple scaling masked by finite-size effects provides a plausible description for all known numerical results on DLA. It is by far the most natural and likely description. In Sec. II, intuitive ideas about why DLA should be self-similar will be discussed. Many of these ideas are incorporated in most theories

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of DLA and they exhibit self-consistency. In contrast, there exists no sensible explanation of why and how any of the complex scalings arises. Section III explains the sources of finite-size effects which lead to deviations from simple scaling. Many of them can be traced back to a crossover in the noise magnitude. The consequences of some complex scaling hypotheses are shown to lead to trivially wrong predictions. Section IV summarizes the paper.

II. IDEAL DLA GROWTH

Self-similarity of DLA often refers to the fact that large clusters of various sizes are statistically similar after proper rescaling. The rationale for the self-similarity can be appreciated intuitively as follows. In two very large clusters of different sizes, the main branches are all composed of similar hierarchies of sidebranches. Assume that the clusters start with similar overall shapes. Since the Laplacian equation is scale invariant and the Laplacian potential depends very much on the overall shape of the clusters instead of the fine details, the potentials for the clusters are similar. Therefore, the evolution of the envelopes of the clusters is similar and the similarity between the two clusters can be sustained. The criterion for the similarity is

$$R_g \gg a \tag{2}$$

where R_g is the radius of gyration of the cluster and a is the particle diameter. For smaller clusters, the evolution will be appreciably different since the Laplacian potential does admit some influences from the length scale of the constituent particles. As the size increases, the difference narrows down and vanishes in the limit when all the main branches considered have practically the same structure of being composed of sidebranches of infinite levels. The dynamics of the coarse-grained aggregate can be considered as being driven to a fixed point. The existence of this fixed point, though not proven, is supported or assumed in several theoretical investigations [3–5].

Although clusters of various sizes are similar, a branch of an aggregate is obviously very different from the whole cluster. This is due to the difference in the environment of growth. While a branch is surrounded by many neighboring branches, the whole cluster grows in an open region and thus very different geometries result. The self-similarity only holds in a local sense. Specifically, segments of various sizes are similar statistically after rescaling if the size l of each of them is in the range

$$R_g \gg l \gg a. \tag{3}$$

In this regime, all segments grow in the same environment formed by the practically infinite medium of the cluster itself. The global boundary conditions are not able to induce influences on the geometry of the Laplacian potential at this much smaller length scale. Homogeneity thus results as well. Local segments anywhere are similar statistically due to similar growth environments, although the growth rate can be very different. However, one might wonder why the center of the cluster is obviously different from any other segments and seems to contradict this local self-similarity hypothesis. The resolution lies in the fact that the probability of sampling the center of the cluster is essentially zero when Eq. (3) holds. Nearly every local part of the cluster grows in the infinite medium of itself and is similar statistically.

The two conditions in Eqs. (2) and (3) for the two manifestations of self-similarity must not be confused. In both cases, the self-similarity is based on the fact that the microscopic details do not influence the growth on longer length scales. This is the reason why DLA characterizes a robust universality class and a large variety of experiments generate DLA-like patterns. Examples include wetting fluid displacement in porous media, growth of hungry bacteria, dendritic growth, electrodeposition, etc. [2]. Similar patterns can also be generated on the computer using algorithms such as the random walker method, solution of the Laplace equation [2], integration of moving boundary equations [15], and integration of partial differential equations with noise. On the other hand, the local self-similarity condition in Eq. (3) guarantees that the local geometry is independent of the global boundary conditions. Therefore, at appropriate length scales, a small segment of DLA in radial geometry is statistically indistinguishable from that of, for example, diffusion-limit deposition.

III. DEVIATIONS FROM SIMPLE SCALING

Numerous simulations show that DLA only follows self-similarity approximately. We will discuss several most widely studied deviations from simple scaling in view of finite-size effects. The discussion is arranged to highlight the relationship among the deviations in various measurements. Some of them are in fact interpreted by many authors as complex scaling instead of finite-size effects. I will attempt to point out the fallacy of those views.

A. Noise; roughness of branch backbones

The decrease in the relative noise magnitude for increasing cluster size is an important finite-size effect which also leads to abnormality in many other measurements. Noise in DLA is due to the randomness in the arrival of walkers. At coarsened scales, it manifests itself as fluctuations in the duration of growth of coarse-grained units. It can be characterized quantitatively by the variance in the growth duration. Using a real space renormalization procedure, Barker and Ball [12] found that upon coarsening the normalized noise magnitude converges to a finite fixed point about two orders of magnitude smaller than the bare one. This decrease is due to the effects of averaging in coarse-grained units. However, the noise does not vanish even asymptotically since the instability in Laplacian growth amplifies microscopic noise. The fixed point corresponds to a compromise between the averaging effects and the instability.

One immediate consequence of the decrease in the noise magnitude is the smoother branches in larger clusters as is evident from visual inspection. The smoothness of a branch may be characterized by the width of its backbone. Ossadnik [13] found numerically that the ratio between the width w and the length l of the backbone of a branch, both defined using an inertia tensor, follows

$$w/l \propto l^{-\eta} \tag{4}$$

where $\eta \simeq 0.21$. The longer branches are thus comparatively thinner because of less wiggling and this is a manifestation of the less noisy growth dynamics. The decrease is not expected to last for much larger clusters. In fact, the effective value of η diminishes for larger branches and Ossadnik inferred from the data that η may approach 0 asymptotically.

In fact, Ossadnik's view that $\eta \to 0$ as $l \to \infty$ is necessarily true for noisy growth. If η remains finite, we have $w/l \to 0$ meaning that all the rescaled backbones are perfectly straight lines asymptotically, which is impossible. Therefore the decrease of w/l for increasing cluster size is a finite-size effect. Asymptotically, $w \propto l$ which is consistent with the self-similarity of DLA.

B. Branching angle

Another important visually observable finite-size effect appears in the angle subtended between a branch and its sidebranch. Ossadnik [13] found numerically that the average subtending angle ϕ_n as a function of the branch order n satisfies

$$\phi_n - 38.4^\circ \propto 0.53^n. \tag{5}$$

The angle decreases from $\phi_0 \simeq 82^\circ$ for the $n = 0$ elementary branches and has nearly converged to $\phi_6 \simeq 38.4^\circ$ for the much longer sixth order branches in clusters of a million particles. The rather large angle 82° for the shortest branches is mainly a consequence of the finite-size effect of the noise magnitude. It is probably due to the more intensive wiggling of the branches which tends to squeeze them further apart. As the noise magnitude decreases at longer length scales, the attraction of the Laplacian field prevails. The asymptotic angle 38.4° compromises between the tendency to growth towards the local flux, which narrows down the angle, and the mutual screening between the sidebranch and its parent, which generates effective repulsion. The convergence to this angle is slow but convincing. It is an important example that a finite-size effect for DLA persists to very large clusters.

The branches investigated are much smaller than R_g. They are roughly in the regime of Eq. (3). Therefore the asymptotic branch subtending angle 38.4° should be independent of the overall boundary conditions and should be the same for diffusion-limited deposition, for instance. This should be compared with the angle subtended by the main branches in DLA in radial geometry, which does not have any counterpart in the deposition case.

C. Width of growing region

Plischke and Racz [6] investigated the region where growth actively occurs for DLA in radial geometry. They found that the mean radius of growth $\bar{R}$ scales as

$$\bar{R} \sim N^{1/D} \tag{6}$$

where N and D are respectively the cluster mass and the fractal dimension. It means that $\bar{R} \sim R_g$ as would be expected from self-similarity.

More importantly, they also computed the width ξ of the active zone defined as the standard deviation of the deposition radius. They found, however, that ξ increases more slowly than proportionately to R_g. They suggested a scaling form

$$\xi \sim N^{\nu'} \quad \text{with} \quad \nu' < \nu = 1/D. \tag{7}$$

This hypothesis implies that the scaling of DLA cannot be characterized by one single exponent and moreover self-similarity is violated. Meakin and Sander [16] subsequently suggested that $\nu' < \nu$ in Eq. (7) is just a finite-size effect and for much larger N the expected relation

$$\xi \sim N^{\nu} \tag{8}$$

which characterizes self-similarity will be restored. More recently, it was suggested that $\nu' = \nu$ but there exists a logarithmic correction so that [7]

$$\xi \sim N^{\nu}/[\ln(N)]^{\beta}. \tag{9}$$

Of the three candidates, Eqs. (7) and (9) both imply that $\xi/R_g \to 0$ asymptotically while Eq. (8) leads to a finite asymptotic value of ξ/R_g. From Ossadnik's [17] recent large scale simulation, ξ/R_g only decreases from 0.225 to 0.195 for N ranging from 20 000 to 500 000. The result does not resolve whether it will converge to 0.

Nevertheless, it is easy to see theoretically that ξ/R_g cannot approach zero so that Eqs. (7) and (9) do not hold asymptotically. This is because it implies that for a very large but rescaled cluster the growing region has zero width. As a result, growth only occurs on a one-dimensional circle. It actually implies that the rescaled aggregate is a dense circular disk. This is because any fjords of finite width would allow penetration of the walkers and lead to a nonzero rescaled width of the growing region. The aggregate is thus macroscopically a continuum with microscopic pores. The pores play little role in dictating the evolution of the continuum in Laplacian growth. According to the Mullins-Sekerka instability [15], even minute fluctuations are amplified so that growth with a perfectly circular front is highly unstable. Therefore, ξ/R_g has to converge to a finite value and Eq. (8) should be the correct asymptotical description.

The decrease of the rescaled width of the active zone is thus simply a finite-size effect. It is mainly due to the weaker noise for larger clusters. In large clusters, there are less fluctuations in the length of the main branches and thus a narrower active zone. When the noise magnitude stabilizes at some very large size, the rescaled width of the growth zone should converge.

D. Radial density

Tolman and Meakin [14] computed the number of particles $N(r)$ within a radial distance r from the center of DLA clusters. Inside the fully grown region, one would expect from self-similarity that

$$N(r) \sim r^{\gamma} \tag{10}$$

with $\gamma = D$. However, using clusters of a million particles each, the effective value of the exponent γ was found to rise from 1.655 for $r \simeq 7$ to a maximum of 1.705 for $r \simeq 665 \simeq 0.44R_g$. Tolman and Meakin suggest that $\gamma = D$ in the limit $R_g \to \infty$, $r \to \infty$ and the difference between D and γ obtained is simply a finite-size effect.

This abnormality is a direct consequence of the finite-size effect in the width of the growing region. We consider the radial density

$$\rho(r) = dN(r)/dr \sim r^{\gamma-1}. \tag{11}$$

The density of an infinite cluster is related to the radial growth probability distribution $P(r,N)$ by

$$\rho(r) = \int_1^{\infty} P(r,N)dN. \tag{12}$$

Inserting any sensible forms of $P(r,N)$ [17] with narrowing rescaled width given by either Eq. (7) or (9), $\rho(r)$ and hence γ can be estimated numerically. [Note that although Eqs. (7) and (9) do not hold asymptotically they are a good description for medium cluster size.] The finite-size effects in γ obtained by Tolman and Meakin can readily be reproduced qualitatively. For much larger clusters, no matter whether the rescaled width converges to a finite value or zero, $\gamma = D$ as expected by Tolman and Meakin. Alternatively, this finite-size effect can also be understood intuitively by considering the mass redistribution due to the narrowing of the rescaled width of the active zone.

E. Multiscaling

Coniglio and Zannetti [7] proposed a multiscaling description for the radial density $\rho(r, R_g)$ of DLA clusters. They suggested

$$\rho(r, R_g) = r^{D(x)-1}A(x) \tag{13}$$

where $x = r/R_g$. Self-similarity implies that $D(x) \equiv D$ for all x. However, a slight dependence of $D(x)$ on x was observed in simulations.

Besides being motivated by numerical results, multiscaling was suggested as a consequence of the abnormality of the scaling behavior of the width ξ of the growing zone. By assuming the radial growth probability distribution to be Gaussian with a width given by Eq. (9) for $\beta = 2$, the radial growth distribution of the rescaled cluster is given by

$$P(x,N) = \frac{1}{\sqrt{2\pi(\xi/R_g)^2}} N^{-\phi(x)} \tag{14}$$

where $\phi(x) = c(x/\bar{x} - 1)^2$, $\bar{x} = \bar{R}/R_g$, and $\bar{R}$ is the mean radius of growth. Since the exponent $\phi(x)$ depends on x, Coniglio and Zannetti neglected the N dependence of the ξ^{-1} factor and claimed that $P(x,N)$ scales differently at every x with respect to N. They concluded that the growth probability follows multiscaling and so may the density of the aggregate.

However, more careful examination of Eq. (14) shows that there is no multiscaling for the growth distribution asymptotically even if Eq. (9) is true. This is because in the limit $N \to \infty$ Eq. (9) actually implies that the distribution converges to the δ function $P(x,N) = \delta(x - \bar{x})$. The multiscaling Coniglio and Zannetti suggested turns out to refer in general to the tail of the δ function for $x \neq \bar{x}$ where no growth occurs. Practically all growth takes place at $\bar{x}$ and it is easy to show that the resulting aggregate density follows simple scaling. Therefore, no matter what one believes about the width of the growing zone, the apparent multiscaling is only a finite-size effect.

The apparent multiscaling can be explained easily by other finite-size effects. For small x so that $r \ll R_g$, growth has completed and $D(x)$ defined in Eq. (13) reduces to γ defined in Eq. (10) for small r. In this regime, Tolman and Meakin [14] found $\gamma \simeq 1.655$ which matches $D(x) \simeq 1.65$ obtained by Ossadnik [18] using million-particle clusters. However, $\gamma < D$ is only a finite-size effect (Sec. III D) and therefore, for small x in the limit of large clusters, $D(x) = \gamma \to D$. For $x \gtrsim 1.5$, Ossadnik [18] found a significant drop of $D(x)$. The point $x \simeq 1.5$ coincides roughly with the mean radial position $\bar{R}$ of growth [16]. For $r > \bar{R}$, narrowing of the rescaled width of the active zone reduces growth and leads to a decrease of the local aggregate density. Therefore $D(x)$ is smaller than D for large x. However, this is again a finite-size effect and $D(x) \to D$ when the width of the active zone has stabilized.

F. ϵ-neighborhood filling ratio

Mandelbrot [9] applied an ϵ-neighborhood analysis to study the lacunarity of DLA which relates to the degree of porosity. In this analysis, every particle in the aggregate is covered with a disk of radius ϵ and the filling ratio which is the fraction of the area occupied by the cover is computed. The investigation was done on circular segments with various radii cut from the fully grown region of a cluster of 14.8 million particles. Using values of ϵ proportional to the radii of the segments, Mandelbrot found a dramatic increase in the filling ratio as the segment size increases. He suggested that the filling ratio may either converge to a constant smaller than 1, in which case DLA becomes self-similar again, or it may approach 1 so that DLA is macroscopically compact asymptotically.

The possibility that the filling ratio may approach 1 is particularly controversial. The overall geometry of such a cluster after rescaling has never been discussed. In fact, it is extremely difficult, if not impossible to imagine a plausible picture. Models like Eden growth [2] can easily generate compact aggregates since growth takes place even behind the growth front. However, due to screening effects in DLA, filling up a hole or even a fjord in the aggregate is impossible. As a result, to generate a compact aggregate, it seems that there needs to be a macroscopically smooth growth front separating the macroscopically dense from the completely empty regions. The aggregate is thus a dense medium with a smooth boundary fairly similar to the case discussed in Sec. III C. Again, the instability of Laplacian growth would contradict the smoothness of the boundary. Therefore this is not a physically plausible picture and the increase of the filling ratio should just be a finite-size effect and it will converge to some constant smaller than 1.

Besides quantitative computation, it is in fact possible to observe visually the increase in the density of rescaled clusters. It is easy to see that both the number of main branches (Sec. III G) and the density of sidebranches increases. This again results from the reduction of the noise magnitude. The reason why a branch can dominate and screen other branches emanating from the same point is usually because of some initial fluctuations which are subsequently amplified by the dynamics [4]. A less noisy environment allows more branches to be equally competitive and survive for longer. As a result, the branch density and thus the fill ratio increase. However, the increment stops when the noise magnitude stabilizes.

G. Number of main branches

Mandelbrot [9] found that as the cluster size increases the separation between the main branches in the rescaled clusters decreases and therefore the number of main branches effectively increases. There is another investigation leading to the same conclusion indirectly: Ossadnik [17] found that the radial growth probability distribution $P(r)$ is not precisely a Gaussian [6] but has a power law tail $P(r) \sim r^{\alpha}$ for $r \ll R_g$. The effective exponent α increases slowly with N and attains a value of about 8.5 for $N = 500\,000$. Idealizing a fjord between two main branches as a cone with an angle θ, the growth probability at r is proportional to $r^{180^\circ/\theta}$. Therefore, the effective angle θ can be estimated as $\theta \simeq 180^\circ/\alpha \simeq 21^\circ$, which is a fairly reasonable value. The increase of α for increasing cluster size thus indicates a decrease in the angle subtended between the main branches and an increase in their number. Both studies cannot resolve whether there will be an infinite number of main branches asymptotically.

Derrida and Hakim [19] studied a needle model of Laplacian growth which indicates that a symmetric pattern with n branches is unstable for $n > 6$. Inferring from their result, it is hard to believe how DLA can maintain growth with many more than six main branches. In addition, an infinite number of branches also corresponds to macroscopically compact rescaled clusters, which is very unlikely as discussed in Sec. III F. The increase in the number of the branches in view of finite-size effects was also explained already in Sec. III F.

H. Multifractal growth probability measure

The multifractal scaling of the growth probability measure is one of the most controversial properties of DLA [2]. Numerical results on clusters of up to 50 000 particles indicate that the $f(\alpha)$ curves for various cluster sizes do not collapse as would be expected for multifractal measures. Several suggestions for the asymptotic scaling of the growth probability have been made. In particular, three mutually incompatible suggestions predict that the minimum growth probability p_{min} follows, respectively [2],

$$p_{min} \sim e^{-aR_g}, \tag{15}$$

$$p_{min} \sim e^{-b(\ln R_g)^c}, \tag{16}$$

$$p_{min} \sim R_g^{-\alpha_{max}}, \tag{17}$$

where a, b, c, and α_{max} are constants.

Equation (15) was motivated by the observation that DLA has long fjords. It can be derived by assuming that the length of the fjords is proportional to R_g while the width is a constant. This suggestion is in fact impossible because it predicts that the average angle ϕ subtended between sidebranches and their parent branches approaches zero. This is in direct contradiction with Ossadnik's numerical result [13] (Sec. III B).

Equation (16) is most consistent with the numerical findings. It follows from a "hierarchical wedge" model, which assumes an *ad hoc* relationship between the width of the openings and the length of the fjords [20]. However, similarly to Eq. (15), it leads to a zero average angle between branches and should be ruled out.

The only alternative which has a sensible geometrical realization is Eq. (17). It results from the self-similarity assumption of DLA. When a walker arrives at a distance l from the cluster with $R_g \gg l \gg a$, where a is the particle diameter, the environment and the Laplacian potential are statistically self-similar and independent of both the microscopic details and the overall boundary conditions (Sec. II). In the asymptotic limit, the walker has to pass through an infinite number of levels of such self-similar environments and this process dominates the growth probability distribution. The probability of growth at any point equals the product of an infinite number of factors each of which corresponds to the probability of, for example, going towards somewhere deeper into a fjord, or entering another fjord at the next level in the self-similar regions. The factor due to the steps at $l > R_g$ and $l \simeq a$ can be neglected. This multiplicative cascade of the probability leads to the multifractal distribution and in particular Eq. (17).

However, self-similarity does not hold accurately for the clusters with 50 000 particles being investigated. In particular, the finite-size effect on the angle ϕ subtended between branches has strong impact on the harmonic measure. Similarly to the discussion in Sec. III G, if we approximate a fjord as a cone of angle $c\phi$ where c

is a constant slightly smaller than 1 to account for the finite thickness of the branches, the growth probability at a distance r from the tip of the cone is proportional to $r^{180°/c\phi}$. There is a very sensitive exponential dependence of the probability on ϕ. According to Eq. (5), ϕ decreases for larger clusters and thus the growth probability inside a fjord will be much smaller than that expected from the self-similarity. Therefore p_{min} decreases much faster with respect to the cluster size than predicted by Eq. (17), although the equation should hold accurately asymptotically.

There have been two major methods to compute the growth probability. One is the numerical solution of the Laplacian equation using relaxation methods. In practice, one has to project the cluster onto a grid which has a resolution strongly limited by the memory available in the computer. This leads to significant systematic errors which cannot be estimated easily. Another method is a Monte Carlo method in which one launches probing walkers and estimates the growth probability according to where they hit. However, the region with low growth probability cannot be probed.

To resolve the controversy numerically, one needs to compute the growth probability measure for significantly larger clusters with good resolution and accuracy. This should be possible using a Monte Carlo method with an importance sampling algorithm called the von Neumann splitting, which is often used in particle physics problems [21]. In this approach, in the rare event that a walker enters a seldom visited region, the walker is split into m similar independent walkers each carrying $1/m$ of the original weight. Those particles can be further split again if they arrive at some even more rarely visited places. In general, if the walker weight can be maintained so that it is roughly proportional to the local particle probability flux, every part of the cluster will be sampled rather evenly. Designing an algorithm to achieve the desired splitting is tricky but possible. Using this method, the present author and collaborators have computed successfully the harmonic measure of a cluster of 100 000 particles. A very wide range of growth probabilities was measured. The result is practically unbiased as no unnecessary gridding was done. The memory required is only slightly more than that used to generate the cluster. Clusters of 130 million particles have been generated [22]. Given comparable hardware, we see no technical problem in solving the harmonic measure for clusters of 100 million particles.

I. Geometrical multifractal

Vicsek, Family, and Meakin [8] suggested that the local density of DLA might scale like a multifractal measure. They examined numerically the scaling relation

$$\left\langle \left(\frac{n(r)}{N}\right)^{q-1} \right\rangle \sim \left(\frac{r}{R_g}\right)^{(q-1)D_q} \tag{18}$$

where $n(r)$ is the number of particles in a box of size r centered at a particle in the cluster and the angular brackets denote averaging over all boxes. Homogeneity of DLA should imply $n(r) \sim r^D$ and thus $D_q = D$ for all q. However, they found a slight dependence of D_q on q and concluded that DLA is a geometrical multifractal.

The multifractal growth probability distribution was suggested as a possible reason for the proposed geometrical multifractal behavior [8]. However, no detail was suggested about how this can be possible. In fact, growth probability at a particular region only dictates the local time scale of growth and is not capable of influencing directly the resulting local geometry. It is the local geometry of the Laplacian field which decides the shape of the growing branches. For example, a fast growing region does not necessarily generate more compact branches. There does not seem to be any reason for the existence of any multiplicative cascade process for the distribution of mass.

In addition, Eq. (18) predicts unreasonably large fluctuations in the local density in the asymptotic limit as shown in the following. For $q \to \infty$, the average in Eq. (18) is dominated by the maximum value $n_{max}(r)$ of $n(r)$. Therefore it reduces to

$$\frac{n_{max}(r)}{N} \sim \left(\frac{r}{R_g}\right)^{D_\infty} . \tag{19}$$

Similarly, the minimum value $n_{min}(r)$ of $n(r)$ scales as

$$\frac{n_{min}(r)}{N} \sim \left(\frac{r}{R_g}\right)^{D_{-\infty}} . \tag{20}$$

Dividing the equations gives

$$\Omega(r) = \frac{n_{max}(r)}{n_{min}(r)} \sim \left(\frac{R_g}{r}\right)^{\kappa} \tag{21}$$

where the magnitude of the fluctuation, $\Omega(r)$, is defined as the ratio between the maximum and the minimum density in boxes of size r and $\kappa = D_{-\infty} - D_\infty \simeq D_{-10} - D_{10} \simeq 0.13$ from Ref. [8].

According to Eq. (21), the fluctuation $\Omega(r)$ for any fixed r increases with R_g. It is already rather counterintuitive that the probability distribution of the density of a local region depends dramatically on the overall cluster size. In addition, for any fixed r, no matter how large, $\Omega(r) \to \infty$ as $R_g \to \infty$. However, $n(r)$ ranges at most from 1 to r^2 since there is at least one particle at the center of the box and the particles do not overlap. Therefore $\Omega(r) < r^2$, contradicting the infinite fluctuation predicted.

As a result, the apparent geometrical multifractal behavior must be a finite-size effect. It is mainly due to the larger relative noise for smaller boxes. This can produce appreciably smaller $n_{min}(r)$ and larger $n_{max}(r)$ than expected from simple scaling. This is already sufficient to explain the numerical result $D_{-\infty} > D > D_\infty$ and the D_q for general q crossover between the two values.

IV. SUMMARY

I have discussed briefly intuitive reasons why DLA should follow simple scaling characterized by the fractal dimension alone. Large clusters of various sizes grown in the radial geometry are similar statistically after rescaling. For length scales much smaller than the cluster but larger than the constituent particles, the cluster is self-similar and the geometry is independent of not only the microscopic details but also the global boundary conditions. The growth probability measure is a simple multifractal. All these properties are consistent with all numerical studies after finite-size effects are identified. It is also consistent with several theories of DLA. It is the only self-consistent scenario suggested in any detail for DLA growth. There are numerous objections to the simple scaling of DLA. Many complex scaling hypotheses do show better agreement with numerical results, if finite-size effects are not taken into account. However, all of them are controversial and none are supported by any theoretical or intuitively viable descriptions.

I have also presented simple arguments showing that the following complex scaling hypotheses in fact lead to physically unacceptable predictions. They are anomalous scaling of the width of the growing region, multiscaling of the cluster radial density, infinite drift of the ϵ-neighborhood filling ratio, nonmultifractal scaling of the growth probability measure, and geometrical multifractality. I suggested possible explanations for the apparent complex scalings in view of finite-size effects. Several measurements or observations which are well known to suffer from finite-size effects are explained. They are the noise magnitude, roughness of branch backbones, branch subtending angle, scaling of the radial density, and the number of main branches. Most of the finite-size effects are indeed closely related to each other and the reason for many of them can be traced back to that of the noise magnitude.

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